# Quantum Cryptography in Algorithmica

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Makrand Sinha, Avishay Tal

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# Introduction

| | |
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| **Algorithmica** | $\mathsf{P} = \mathsf{NP}$ |
| **Heuristica** | $\mathsf{P} \neq \mathsf{NP}$ but $\mathsf{DistNP} \subseteq \mathsf{AvgP}$ |
| **Pessiland** | $\mathsf{DistNP} \not\subseteq \mathsf{AvgP}$ but $\nexists$ OWFs |
| **Minicrypt** | $\exists$ OWFs but $\nexists$ PKE |
| **Cryptomania** | $\exists$ PKE |

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$f : \{0,1\}^n \to \{0,1\}^m$ is *one-way* if:

- $f$ efficiently computable
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**Necessary** and **sufficient** for lots of classical cryptography

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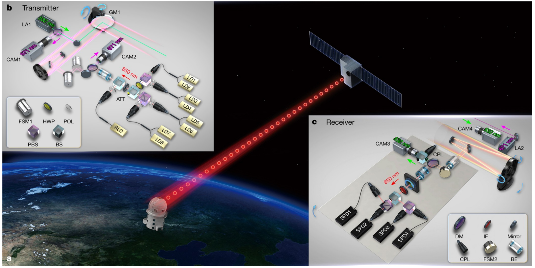


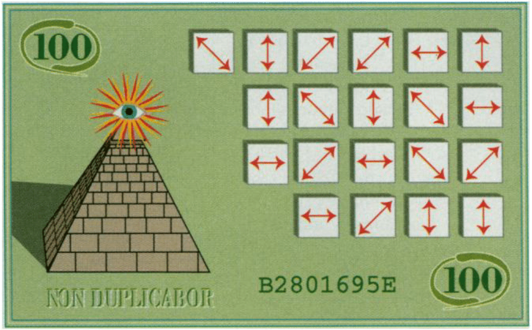

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## Definition (Ji-Liu-Song 2018)

$\{|\varphi_k\rangle\}_{k\in\{0,1\}^\kappa}$ is *pseudorandom* if:

- Efficient generation of $|\varphi_k\rangle$ given $k \in \{0,1\}^\kappa$
- For all poly-time $\mathcal{A}$ and $T = \text{poly}(\kappa)$:

$$\Pr_{k\sim\{0,1\}^\kappa}\left[\mathcal{A}\left(|\varphi_k\rangle^{\otimes T}\right) = 1\right] - \Pr_{|\psi\rangle\leftarrow\mu_{\text{Haar}}}\left[\mathcal{A}\left(|\psi\rangle^{\otimes T}\right) = 1\right] \leq \text{negl}(\kappa)$$

## Definition (Morimae-Yamakawa 2022)

$\{|\varphi_k\rangle\}_{k\in\{0,1\}^\kappa}$ is *single-copy pseudorandom* if:

- $\kappa < n$, where $n = \#$ qubits
- Efficient generation of $|\varphi_k\rangle$ given $k \in \{0,1\}^\kappa$
- For all poly-time $\mathcal{A}$:

$$\Pr_{k\sim\{0,1\}^\kappa}[\mathcal{A}(|\varphi_k\rangle) = 1] - \Pr_{|\psi\rangle\leftarrow\mu_{\text{Haar}}}[\mathcal{A}(|\psi\rangle) = 1] \leq \text{negl}(\kappa)$$

- **Suffice** for commitments, signatures, multiparty computation, zero-knowledge... [Morimae-Yamakawa 2022, Ananth-Qian-Yuen 2022]
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**Theorem [K. 2021]**

There is a **quantum oracle** $\mathcal{O}$ such that:

1. $\mathsf{BQP}^{\mathcal{O}} = \mathsf{QMA}^{\mathcal{O}}$, and
2. PRSs exist relative to $\mathcal{O}$

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Limitations:

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- Quantum oracles are weak
- Not real-world instantiable

# This Work

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(1) **Suffices** for single-copy PRSs

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(3) Is **independent** of P vs NP in the black box setting

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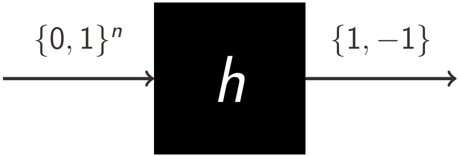


Given $h$, decide if:

(1) $h$ uniformly random

(2) $\exists k$: $h$ correlated with $\hat{f}_k \cdot g_k$

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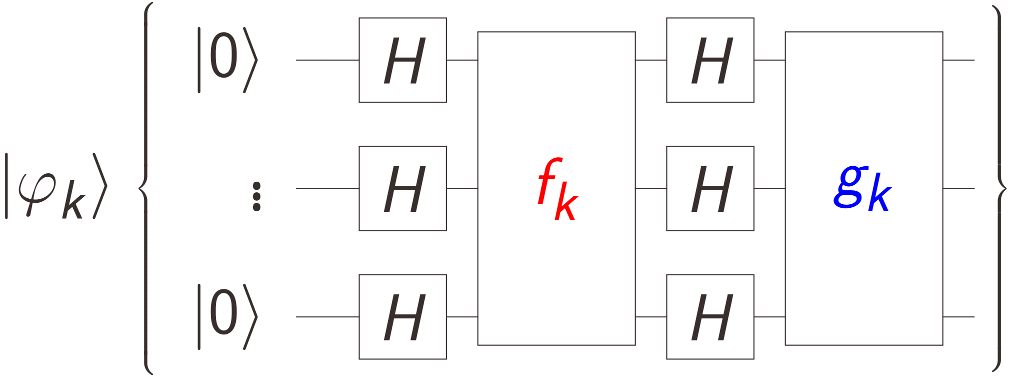

## Forrelation [Aaronson 2009]

Given $f, g : \{0,1\}^n \to \{1,-1\}$, decide if:

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- Forrelation $\in$ BQP [Aaronson 2009]
- Forrelation $\notin$ PH [Raz-Tal 2018]
- OR $\circ$ Forrelation $\notin$ $\text{BQP}^{\text{PH}}$ [Aaronson-Ingram-**K.** 2022]

H$_1$: $|\varphi_k\rangle$

H$_2$: $|\Phi_h\rangle := \frac{1}{\sqrt{2^n}} \sum_x h(x)|x\rangle$ for $h$ correlated w/ $\hat{f}_k \cdot g_k$

H$_3$: $|\Phi_h\rangle$ for $h$ uniform

H$_4$: $|\psi\rangle$ Haar-random

# Open Problems

# Multi-copy security? True under a conjecture about $t$-Forrelation

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Oracle where $\mathsf{P} = \mathsf{QMA}$ but PRSs exist?

Do single-copy PRSs imply $\mathsf{P} \neq \mathsf{PSPACE}$?

# William Kretschmer

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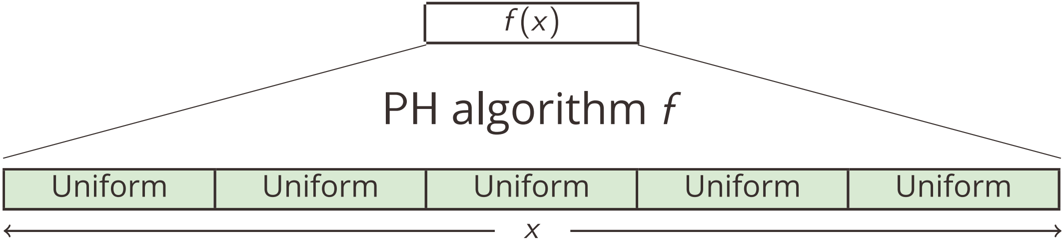
Computer Science

- Goal: OR $\circ$ Forrelation $\notin \mathsf{BQP}^{\mathsf{PH}}$
- Idea: PH can't be "sensitive" to a single Forrelated block



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